

An Alternative Formulation of Hall Effect and Quantum Phases in Noncommutative Space

Ömer F. Dayı^{a,b}, and Barış Yapışkan^c

^a*Physics Department, Faculty of Science and Letters, Istanbul Technical University,
TR-34469, Maslak–Istanbul, Turkey*

^b*Feza Gürsey Institute, P.O. Box 6, TR-34684, Çengelköy, Istanbul, Turkey*

^c*Physics Department, Faculty of Science and Letters, Mimar Sinan Fine Arts University,
Çırağan Cad. Çiğdem Sok. No:1, TR-34349, Beşiktaş –Istanbul, Turkey*

Abstract

A recent method of constructing quantum mechanics in noncommutative coordinates, alternative to implying noncommutativity by means of star product is discussed. Within this approach we study Hall effect as well as quantum phases in noncommutative coordinates. The θ -deformed phases which we obtain are velocity independent.

E-mail addresses: dayi@itu.edu.tr and dayi@gursey.gov.tr; barisy@gursey.gov.tr

1 Introduction

Formulation of quantum mechanics avoiding operators was given at the beginning of quantum era by Weyl–Wigner–Groenewold–Moyal (WWGM)[1] as the \hbar -deformation of classical mechanics: Observables are functions taking values in the classical phase space (P_I, Q_I) whose combinations are constructed through the star product

$$\star = \exp \left[\frac{i\hbar}{2} \left(\overleftarrow{\partial} \frac{\overrightarrow{\partial}}{\partial Q^I} \frac{\overrightarrow{\partial}}{\partial P_I} - \frac{\overleftarrow{\partial}}{\partial P_I} \frac{\overrightarrow{\partial}}{\partial Q^I} \right) \right], \quad (1)$$

where $\overleftarrow{\partial}$ ($\overrightarrow{\partial}$) indicates that the derivative is applied to the left (right). Although the WWGM formulation of quantum mechanics has some shortcomings, like the lack of positive definite probabilities, it turned out to be crucial to establish quantum mechanics in noncommutative coordinates: One treats coordinates as commutative and imply the noncommutativity by the star product given with an antisymmetric, constant deformation parameter θ_{IJ} as

$$\star_\theta \equiv \exp \left[\frac{i}{2} \theta_{IJ} \frac{\overleftarrow{\partial}}{\partial Q^I} \frac{\overrightarrow{\partial}}{\partial Q^J} \right]. \quad (2)$$

θ -deformed Hamiltonian systems are built inserting the star product (2) between the bilinear or higher terms appearing in the original quantum Hamiltonians. Indeed, this is equivalent to the shift of coordinates

$$Q_I \rightarrow Q_I - \frac{1}{2\hbar} \theta_{IJ} P_J^{op}, \quad (3)$$

where $P_I^{op} = -i\hbar \frac{\partial}{\partial Q^I}$ is used. As it is obvious when the original Hamiltonian does not possess any coordinate dependence this procedure will not furnish any θ -deformation. Especially, it is not adequate to consider spin degrees of freedom.

There is an alternative method of defining quantum Hamiltonians in noncommutative spaces[2] as far as in the starting Hamiltonian there exist terms which can be interpreted as minimally coupled gauge fields which may be either functions or matrix valued. In this work we would like to discuss this alternative procedure of obtaining dynamical systems in noncommuting coordinates and apply it to some interesting physical systems where the Hamiltonians are functions or matrices.

Employing the star product (2) or the equivalent shift (3), diverse physical systems like Hall effect in noncommuting coordinates[3] and quantum phases in noncommutative spaces[3, 4, 5, 6, 7, 8] were considered. In all these dynamical problems there are some external fields which can be interpreted as gauge fields interacting with the particles in terms of the minimal coupling procedure. Within the alternative method we first show that Hall effect in noncommutative coordinates can be formulated where resulting Hall conductivity is θ -deformed or not, depending on the realization adopted. Then, we apply the new deformation procedure to obtain velocity independent formulations of Aharonov–Bohm (AB)[9], Aharonov–Casher (AC)[10], He–McKellar–Wilkins (HMW)[11] and Anandan[12, 13] phases in noncommutative coordinates. Most of the earlier formulations yielded velocity dependent quantum phases in noncommutative spaces, in spite of the fact that the distinguished property of the original phases is their independence from the velocity of scattered particles. We presented a unified formulation of the θ -deformed quantum phases yielding velocity dependent terms. The difference between them and our method is clarified. In the last section we discuss the results obtained and their consequences. In particular we discuss how to select the suitable realization.

2 The Alternative θ -deformation of Quantum Mechanics

Generating classical mechanics as the $\hbar \rightarrow 0$ limit of quantum mechanics can be best perceived by the WWGM method. Let (P_I, Q_I) denote the classical variables corresponding to the quantum phase space variables (P_I^{op}, Q_I^{op}) ; $I = 1, \dots, M$. Multiplication of observables in the former space is given by the usual operator product, however in the latter the star product (1) is employed to carry out multiplications. In the WWGM approach, to imitate quantum commutators one introduces the Moyal brackets

$$[f(P, Q), g(P, Q)]_\star \equiv f(P, Q) \star g(P, Q) - g(P, Q) \star f(P, Q),$$

where the observables $f(P, Q)$ and $g(P, Q)$ are some functions. Hence the classical limit of the commutators is equivalent to

$$\lim_{\hbar \rightarrow 0} \frac{-i}{\hbar} [f(P, Q), g(P, Q)]_\star = \{f(P, Q), g(P, Q)\} \equiv \frac{\partial f}{\partial Q^I} \frac{\partial g}{\partial P_I} - \frac{\partial f}{\partial P_I} \frac{\partial g}{\partial Q^I}, \quad (4)$$

which is the Poisson bracket. When the observables are matrices whose elements are $M_{kl}(P, Q)$ and $N_{kl}(P, Q)$, the Moyal bracket is

$$([M(P, Q), N(P, Q)]_\star)_{kl} = M_{km}(P, Q) \star N_{ml}(P, Q) - N_{km}(P, Q) \star M_{ml}(P, Q). \quad (5)$$

However, its classical limit (4), in addition to the Poisson brackets, yields a singular commutator of matrices. Hence, instead of the classical limit (4) we deal with the “semiclassical” limit defined by retaining the terms up to \hbar of the Moyal bracket (5):

$$\{M(P, Q), N(P, Q)\}_C \equiv \frac{-i}{\hbar} [M, N] + \frac{1}{2} \{M(P, Q), N(P, Q)\} - \frac{1}{2} \{N(P, Q), M(P, Q)\}. \quad (6)$$

The first term is the commutator of matrices, it should not be confused with the quantum mechanical commutator. Obviously, the bracket (6) does not satisfy Jacobi identities. However, we consider a semiclassical approach in which the semiclassical limit is taken after performing multiplication of the observables in terms of the star product. Hence, vanishing of the first two terms of the Moyal bracket relation

$$\frac{-i}{\hbar} (\{M, \{N, L\}_\star\}_\star + \{N, \{L, M\}_\star\}_\star + \{L, \{M, N\}_\star\}_\star) = \mathcal{O}_{-1}(\frac{1}{\hbar}) + \mathcal{O}_0(\hbar^0) + \mathcal{O}_1(\hbar) + \dots$$

should be considered in the semiclassical limit of the Jacobi identity. Indeed, one can show that the semiclassical limit of the Jacobi identity

$$\begin{aligned} \mathcal{O}_{-1}(\frac{1}{\hbar}) + \mathcal{O}_0(\hbar^0) &= -\frac{i}{\hbar} [M, [N, L]] + [M, \{N, L\}] - [M, \{L, N\}] \\ &\quad + \{M, [N, L]\} - \{[N, L], M\} + (\text{cyclic permutations of } M, N, L) = 0 \end{aligned}$$

is satisfied. Moreover, one can observe that the semiclassical limit of the Leibniz rule defined as

$$\{M \star N, L\}_C = \{M, L\}_C \star N + M \star \{N, L\}_C,$$

is also satisfied at the \hbar order. Although, these considerations are essential for a consistent definition of the semiclassical method of matrix observables, in this work we will deal with a quantum phase space algebra obtained somehow utilizing the bracket (6). Obviously, adopted operator realization

of commutators among the phase space variables should satisfy the usual Jacobi identities as we will discuss.

Dynamical systems in noncommutative space can be formulated in terms of the following first order matrix Lagrangian (for the most general case see [14]), by choosing the coordinates as $Q_I = (r_\alpha, p_\alpha)$, where $\alpha, \beta = 1, \dots, d$,

$$L = \dot{r}^\alpha \left[\frac{p_\alpha}{2} \mathbb{I} + \rho A_\alpha(r) \right] - \frac{\dot{p}^\alpha}{2} \mathbb{I} \left[r_\alpha + \frac{\theta_{\alpha\beta}}{\hbar} p^\beta \right] - H_0(r, p). \quad (7)$$

\mathbb{I} denotes the unit matrix possessing the same dimension with the matrix valued gauge field A_α . We denoted the coupling constant as ρ . Although (7) is classical, \hbar is present to furnish the constant, antisymmetric noncommutativity parameter $\theta_{\alpha\beta}$ with the dimension $(\text{length})^2$. The canonical momenta $P_I = (P_r^\alpha = \partial L / \partial \dot{r}_\alpha, P_p^\alpha = \partial L / \partial \dot{p}_\alpha)$, corresponding to the coordinates $Q_I = (r_\alpha, p_\alpha)$, yield the dynamical constraints

$$\begin{aligned} \psi^{1\alpha} &\equiv (P_r^\alpha - \frac{1}{2} p^\alpha) \mathbb{I} - \rho A^\alpha, \\ \psi^{2\alpha} &\equiv (P_p^\alpha + \frac{1}{2} r^\alpha) \mathbb{I} + \frac{\theta^{\alpha\beta}}{\hbar} p_\beta. \end{aligned}$$

They satisfy the semiclassical relations

$$\begin{aligned} \{\psi_\alpha^1, \psi_\beta^1\}_C &= \rho F_{\alpha\beta}, \\ \{\psi_\alpha^2, \psi_\beta^2\}_C &= \frac{\theta_{\alpha\beta}}{\hbar}, \\ \{\psi_\alpha^1, \psi_\beta^2\}_C &= -\delta_{\alpha\beta}, \end{aligned}$$

where we introduced the field strength:

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial r^\alpha} - \frac{\partial A_\alpha}{\partial r^\beta} - \frac{i\rho}{\hbar} [A_\alpha, A_\beta]. \quad (8)$$

We would like to emphasize the fact that commutators appearing in this semiclassical formulation are the ordinary matrix commutators. Obviously, ψ_α^z ; $z = 1, 2$, are second class constraints, so that one can take them into account by introducing the semiclassical Dirac bracket defined as

$$\{M, N\}_{CD} \equiv \{M, N\}_C - \{M, \psi^z\}_C \mathcal{C}_{zz'}^{-1} \{\psi^{z'}, N\}_C, \quad (9)$$

in terms of \mathcal{C}^{-1} which is the inverse of

$$\mathcal{C}_{\alpha\beta}^{zz'} = \{\psi_\alpha^z, \psi_\beta^{z'}\}_C.$$

Therefore, omitting \mathbb{I} on the left hand sides, one can show that the following relations are satisfied,

$$\{r^\alpha, r^\beta\}_{CD} = \frac{\theta^{\alpha\beta}}{\hbar}, \quad (10)$$

$$\{p^\alpha, p^\beta\}_{CD} = \rho F^{\alpha\beta} - \frac{\rho^2}{\hbar} (F\theta F)^{\alpha\beta}, \quad (11)$$

$$\{r^\alpha, p^\beta\}_{CD} = \delta^{\alpha\beta} - \frac{\rho}{\hbar} (\theta F)^{\alpha\beta}, \quad (12)$$

keeping the terms at the first order in θ and at the second order in ρ . We abbreviated $(\theta F)^{\alpha\beta} \equiv \theta^{\alpha\gamma} F_{\gamma}^{\beta}$, $(F\theta F)^{\alpha\beta} \equiv F^{\alpha\gamma} \theta_{\gamma}^{\sigma} F_{\sigma}^{\beta}$.

The semiclassical brackets (6) differ from the Poisson brackets up to commutators of matrices, so that for observables which are not matrices (9) reduce to the ordinary Dirac brackets. Therefore, we can extend the canonical quantization rules to embrace the matrix observables by the substitution of the basic brackets (10)–(12) with the quantum commutators as $\{, \}_{CD} \rightarrow \frac{1}{i\hbar}[,]_q$. Then, we are furnished with the generalized algebra

$$[\hat{r}^{\alpha}, \hat{r}^{\beta}]_q = i\theta^{\alpha\beta}, \quad (13)$$

$$[\hat{p}^{\alpha}, \hat{p}^{\beta}]_q = i\hbar\rho F^{\alpha\beta} - i\rho^2(F\theta F)^{\alpha\beta}, \quad (14)$$

$$[\hat{r}^{\alpha}, \hat{p}^{\beta}]_q = i\hbar\delta^{\alpha\beta} - i\rho(\theta F)^{\alpha\beta}, \quad (15)$$

$$[\hat{p}^{\alpha}, \hat{r}^{\beta}]_q = -i\hbar\delta^{\alpha\beta} + i\rho(F\theta)^{\alpha\beta}. \quad (16)$$

We denoted quantum commutators by the subscript q to distinguish them from matrix commutators. Because of being first order in θ , the right hand sides of (13)–(16) may only possess $\hat{r}_{\alpha}|_{\theta=0} = r_{\alpha}$ dependence. Hence $F_{\alpha\beta}$ is still as in (8). This is the starting point of the alternative method of establishing quantum mechanics in noncommutative coordinates. A realization of the generalized algebra (13)–(16) and a Hamiltonian $H_0(p, r)$ should be provided. Let us deal with the operators

$$\hat{p}_{\alpha} = D_{\alpha} - \frac{\rho}{2\hbar} F_{\alpha\beta} \theta^{\beta\gamma} D_{\gamma}, \quad (17)$$

$$\hat{r}_{\alpha} = r_{\alpha} - \frac{1}{2\hbar} \theta_{\alpha\beta} D^{\beta}, \quad (18)$$

where the covariant derivative is

$$D_{\alpha} = -i\hbar \frac{\partial}{\partial r^{\alpha}} - \rho A_{\alpha}.$$

One can demonstrate that (17) and (18) satisfy the algebra (13)–(16) and the Jacobi identities, as far as the conditions

$$-i\hbar \nabla_{\alpha} F_{\beta\gamma} - \rho[A_{\alpha}, F_{\beta\gamma}] = 0, \quad [F_{\alpha\beta}, F_{\gamma\delta}] = 0 \quad (19)$$

are fulfilled. To illustrate the method let the initial Hamiltonian be $H_0(p) = p^2/2m$. Substituting p with the quantum operator (17) one obtains the θ -deformed Hamiltonian

$$H_0(\hat{p}) \equiv \hat{H}_{nc} = \frac{1}{2m} \left(D_{\alpha} - \frac{\rho}{2\hbar} F_{\alpha\beta} \theta^{\beta\gamma} D_{\gamma} \right)^2. \quad (20)$$

Setting $\theta = 0$ yields the Hamiltonian operator

$$\hat{H} = \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial r_{\alpha}} - \rho A_{\alpha} \right)^2. \quad (21)$$

Therefore, (20) gives the noncommutative dynamics corresponding to the Hamiltonian (21).

Let us present another representation of the algebra (13)–(16) which will be utilized in the subsequent sections. As far as (19) are valid, one can prove that

$$\hat{p}_{\alpha} = -i\hbar \nabla_{\alpha} + \frac{\rho}{2} F_{\alpha\beta} (r^{\beta} + 2i\theta^{\beta\gamma} \nabla_{\gamma}), \quad (22)$$

$$\hat{r}_{\alpha} = r_{\alpha} - \frac{1}{2\hbar} \theta_{\alpha\beta} (-i\hbar \nabla^{\beta} - \frac{\rho}{2} F^{\beta\gamma} r_{\gamma}) \quad (23)$$

constitute another realization of the algebra (13)–(16). It is worth mentioning that in this representation only the gauge invariant field strength $F_{\alpha\beta}$ appears, in contrast to (17)–(18) where the gauge field A_{α} explicitly appears.

3 Hall Effect in Noncommutative Coordinates

We would like to study the Hall effect in noncommutative space in the light of the alternative method prescribed in Section 2, adopting different realizations. This problem was addressed previously in [3] where the Hall conductivity was shown to acquire a θ -deformation factor. However this resulted to the cost of an unnatural overall factor in the definition of electric current. We will show that θ -deformation of the Hall conductivity appears naturally in some realizations. Though in [15] a natural θ -deformed Hall conductivity was achieved, it was within the semiclassical approach of Section 2.

We consider an electron moving on the two-dimensional plane $r_i = (x, y)$ in the presence of the uniform external in-plane electric field E and the uniform external perpendicular magnetic field B . The latter is taken into account by the field strength $F_{ij} = \epsilon_{ij}B$, and fixing $\rho = -e/c$. Hence the generalized algebra (13)–(16) yields

$$[\hat{x}, \hat{y}]_q = i\theta, \quad (24)$$

$$[\hat{p}_i, \hat{p}_j]_q = -\frac{ieB\hbar}{c} \left(1 - \frac{eB\theta}{\hbar c}\right) \epsilon_{ij}, \quad (25)$$

$$[\hat{r}_i, \hat{p}_j]_q = i\hbar \left(1 - \frac{e\theta B}{\hbar c}\right) \delta_{ij}. \quad (26)$$

We choose the electric field to lie in the direction of the x -axis. Thus the Hamiltonian is taken as

$$H = \frac{1}{2m} \hat{p}_i^2 + eE\hat{x}. \quad (27)$$

First consider the realization of (24)–(26) given in (17)–(18) by choosing the symmetric gauge $A_i = \frac{eB}{2c} \epsilon_{ij} r_j$, but ignoring the terms at the e^2/c^2 order:

$$\hat{p}_i^{(1)} = \left(1 - \frac{e\theta B}{2\hbar c}\right) p_i - \frac{eB}{2c} \epsilon_{ij} r^j, \quad (28)$$

$$\hat{r}_i^{(1)} = \left(1 - \frac{e\theta B}{4\hbar c}\right) r_i - \frac{\theta}{2\hbar} \epsilon_{ij} p^j. \quad (29)$$

By plugging (28) and (29) into (27) one obtains the Hamiltonian

$$H^{(1)} = \frac{1}{2m} \left[\left(1 - 2\kappa\right) p_i - \frac{eB}{2c} \epsilon_{ij} r^j \right]^2 + eE(1 - \kappa)x - \frac{eE\theta}{2\hbar} p_y, \quad (30)$$

where we defined $p_i \equiv -i\hbar \nabla_i$ and $\kappa = \frac{e\theta B}{4\hbar c}$. On the other hand keeping the terms at the e^2/c^2 order yields the following realization

$$\hat{p}_i^{(2)} = \left(1 - \frac{e\theta B}{2\hbar c}\right) p_i - \frac{eB}{2c} \left(1 - \frac{e\theta B}{2\hbar c}\right) \epsilon_{ij} r^j, \quad (31)$$

$$\hat{r}_i^{(2)} = \left(1 - \frac{e\theta B}{4\hbar c}\right) r_i - \frac{\theta}{2\hbar} \epsilon_{ij} p^j. \quad (32)$$

When these are substituted in (27), one gets the θ -deformed Hamiltonian as

$$H^{(2)} = \frac{1}{2m} (1 - 2\kappa)^2 \left[p_i - \frac{eB}{2c} \epsilon_{ij} r^j \right]^2 + eE(1 - \kappa)x - \frac{eE\theta}{2\hbar} p_y. \quad (33)$$

Another realization of (24)–(26) can be read from (22)–(23): It does not refer to the explicit form of the vector field,

$$\hat{p}_i^{(3)} = (1 - \frac{e\theta B}{\hbar c})p_i - \frac{eB}{2c}\epsilon_{ij}r^j, \quad (34)$$

$$\hat{r}_i^{(3)} = (1 + \frac{e\theta B}{4\hbar c})r_i - \frac{\theta}{2\hbar}\epsilon_{ij}p^j. \quad (35)$$

Plugging (34) and (35) into (27) will produce the deformed Hamiltonian

$$H^{(3)} = \frac{1}{2m} \left[(1 - 4\kappa)p_i - \frac{eB}{2c}\epsilon_{ij}r^j \right]^2 + eE(1 + \kappa)x - \frac{eE\theta}{2\hbar}p_y. \quad (36)$$

It is not surprising that there exist different θ -deformations of an underlying Hamiltonian. However, as we will argue in Section 5, preferring one to others is possible by adopting an interpretation of the θ -deformation and then specifying the adequate θ -deformed physical quantities derived from the deformed Hamiltonians.

Now we will examine these Hamiltonians in detail. In order to discuss the eigenvalue problem of the Hamiltonians we perform the following change of variables

$$z = x + iy, \quad p_z = \frac{1}{2}(p_x - ip_y).$$

Let us introduce two sets of creation and annihilation operators:

$$\begin{aligned} b &= 2i\gamma p_z + \frac{eB}{2c}\beta\bar{z} + \lambda_-; & d &= 2i\gamma p_z - \frac{eB}{2c}\beta\bar{z}, \\ b^\dagger &= -2i\gamma p_{\bar{z}} + \frac{eB}{2c}\beta z + \lambda_-; & d^\dagger &= -2i\gamma p_{\bar{z}} - \frac{eB}{2c}\beta z. \end{aligned}$$

The constant coefficients γ , β and λ_- will be fixed for each Hamiltonian separately. These two mutually commuting sets of operators satisfy the commutation relations

$$[b, b^\dagger] = 2m\hbar\gamma\beta\omega, \quad [d, d^\dagger] = -2m\hbar\gamma\beta\omega,$$

where $\omega = eB/mc$ is the cyclotron frequency. Each of the Hamiltonians (30), (33) and (36) can be written in terms of creation and annihilation operators in the form

$$H = \frac{1}{4m}(bb^\dagger + b^\dagger b) - \frac{\lambda_+}{2m}(d + d^\dagger) - \frac{\lambda_-^2}{2m}, \quad (37)$$

where the constant λ_+ is also going to be fixed for each Hamiltonian separately. The natural definition of the current operator is

$$J_i = \frac{ie\rho_e}{\hbar}[H, r_i] = \frac{e\rho_e\gamma}{m}(\gamma p_i - \beta\frac{eB}{2c}\epsilon_{ij}r^j + a_i), \quad (38)$$

where ρ_e stands for electron density and $a_i = (0, -\frac{emE\theta}{2\hbar\gamma})$. The expectation value of the current operator $\langle J_i \rangle$ can be calculated with respect to the eigenstates of the Hamiltonian (37) given as[3]

$$|n, \alpha, \theta\rangle = \frac{1}{\sqrt{(2m\hbar\gamma\beta)^n n!}} \exp\{i(\alpha y + \frac{m\gamma\beta}{2\hbar}xy)\}(b^\dagger)^n |0\rangle, \quad n = 0, 1, 2, \dots, \quad \alpha \in \mathbb{R}.$$

By definition $b|0\rangle = 0$. Once the current operator is obtained in terms of creation and annihilation operators, the calculation is straightforward. Indeed, one can easily show that expectation value of the x -component of current vanishes:

$$\langle J_x \rangle = \langle n, \alpha, \theta | \frac{e\rho_e\gamma}{2im} (b - b^\dagger) | n, \alpha, \theta \rangle = 0. \quad (39)$$

On the other hand expectation value of the y -component leads to the Hall conductivity σ_H :

$$\begin{aligned} \langle J_y \rangle &= \frac{e\rho_e\gamma}{m} \langle n, \alpha, \theta | \left(\frac{b + b^\dagger}{2} - \lambda_- - \frac{emE\theta}{2\hbar\gamma} \right) | n, \alpha, \theta \rangle \\ &= -\frac{e\rho_e}{m} \left(\gamma\lambda_- + \frac{emE\theta}{2\hbar} \right) = \sigma_H E. \end{aligned} \quad (40)$$

Now we will analyze each Hamiltonian separately: The coefficients related to (30), (33) and (36) are presented in Table 1.

	γ	β	λ_+	λ_-
$H^{(1)}$	$1 - 2\kappa$	1	$(1 - \kappa)\lambda + \frac{emE\theta}{4\hbar}$	$(1 - \kappa)\lambda - \frac{emE\theta}{4\hbar}$
$H^{(2)}$	$1 - 2\kappa$	$1 - 2\kappa$	$(1 + 2\kappa)\lambda$	λ
$H^{(3)}$	$1 - 4\kappa$	1	$(1 + \kappa)\lambda + \frac{emE\theta}{4\hbar}$	$(1 + \kappa)\lambda - \frac{emE\theta}{4\hbar}$

Table 1: Coefficients of the diverse realizations in terms of $\kappa = \frac{e\theta B}{4\hbar c}$ and $\lambda = \frac{mcE}{B}$.

For the Hamiltonian (33) the Hall conductivity does not acquire any θ -deformation:

$$\sigma_H^{(2)} = -\frac{\rho_e e c}{B}. \quad (41)$$

Conversely, one can observe that, although their coefficients differ the Hamiltonians (30) and (36) give the same result for the Hall conductivity:

$$\sigma_H^{(1)}(\theta) = \sigma_H^{(3)}(\theta) = -\frac{\rho_e e c}{B} \left(1 - \frac{e\theta B}{2\hbar c} \right). \quad (42)$$

This result is compatible with the one obtained in [3] although the deformation factors do not coincide. However, in [3] θ -deformation results due to a specific choice of overall factor in the definition of current, here the current does not possess any unnatural coefficient in its definition.

There are some interesting features. In the realization (17)-(18) if one does not keep the e^2/c^2 terms Hall conductivity acquires a deformation factor, in contrary to the case where the e^2/c^2 terms are kept. Higher order corrections in the realization sweeps out the θ dependence of the Hall conductivity. Another curious result is the fact that although their structures are different, two Hamiltonians (30) and (36) lead to the same deformation factor for the Hall conductivity. Consequences of obtaining various types of θ -deformations of the Hall conductivity will be discussed in the last section.

4 Quantum Phases in Noncommutative Space

The existing formulations of the quantum phases in noncommutative coordinates can mainly be distinguished by their dependence on momentum eigenvalues: The formulations of [4, 5, 6, 7, 8] depend

on momentum eigenvalues but the ones in [3] and [16] do not possess any momentum dependence as the original quantum phases. Except [16] where a (semi)classical approach was used, all of these formulations implement noncommutativity in terms of the star product (2) or the equivalent coordinate shift (3), however interpretation of the θ dependent terms appearing in Hamiltonians differ. Because of adopting the interpretation of [3], the θ -deformed phases which we obtain are also momentum independent. This issue will be clarified at the end of this section. First, we would like to discuss the existing formulations. Although in [4, 5, 6, 7, 8] different phases are considered we will show that they can be formulated in a unified manner. The starting Hamiltonian operator is

$$H = \frac{1}{2m} (p_\alpha - \rho A_\alpha(r))^2, \quad (43)$$

where ρ is a constant and the configuration is chosen such that the scalar potential term vanishes. One implements noncommutativity by the shift

$$r_\alpha \rightarrow r_\alpha - \frac{1}{2\hbar} \theta_{\alpha\beta} p^\beta = r_\alpha - \frac{1}{2\hbar} \theta_{\alpha\beta} \left(\hbar k^\beta + \rho A^\beta(r) \right), \quad (44)$$

where k_α is the eigenvalue of the kinetic momentum operator:

$$(p_\alpha - \rho A_\alpha(r)) \psi(r) = \hbar k_\alpha \psi(r). \quad (45)$$

Hence, (43) can be expanded at the first order in θ as

$$H = \frac{1}{2m} \left[p_\alpha - \rho A_\alpha(r) + \frac{\rho}{2\hbar} \theta^{\beta\sigma} (\hbar k_\sigma + \rho A_\sigma(r)) \partial_\beta A_\alpha(r) \right]^2. \quad (46)$$

Identifying,

$$\tilde{A}_\alpha(r, \theta) = A_\alpha(r) - \frac{1}{2\hbar} \theta^{\beta\sigma} (\hbar k_\sigma + \rho A_\sigma(r)) \partial_\beta A_\alpha(r) \quad (47)$$

as the gauge field in noncommutative space, one defines the θ -deformed quantum phase by

$$\Phi(\theta) = \frac{i\rho}{\hbar} \oint \tilde{A}_\alpha(r, \theta) dr^\alpha. \quad (48)$$

Different phases can be considered by choosing the original field A_α appropriately. To study the AB phase on the noncommutative plane let the nonvanishing components of the deformation parameter be

$$\theta_{ij} = \theta \epsilon_{ij},$$

where $i, j = 1, 2$. Moreover, choose $\rho = -e/c$ and an appropriate 3-vector potential \mathbf{A} , whose third component vanishes $A_3 = 0$. Hence, (48) leads to

$$\Phi_{AB}^I(\theta) = -\frac{ie}{\hbar c} \oint \mathbf{A}(r) \cdot d\mathbf{r} - \frac{ime\theta}{2\hbar c} \oint \left[(\mathbf{v} \times \nabla A_i)_3 - \frac{e}{\hbar mc} (\mathbf{A} \times \nabla A_i)_3 \right] dr_i,$$

where $\mathbf{k} = m\mathbf{v}$. This is the deformed phase obtained in [4] and [5].

To formulate the AC, HMW and Anandan phases in noncommutative coordinates we set

$$c\rho\mathbf{A} = \boldsymbol{\mu} \times \mathbf{E} - \mathbf{d} \times \mathbf{B} \quad (49)$$

where $\boldsymbol{\mu}$ and \mathbf{d} are the magnetic and the electric dipole moments which are proportional to the Pauli spin matrices $\boldsymbol{\sigma}$. We deal with the standard configuration where dipole moments are in z -direction and

the external electric and magnetic fields are in the polar radial direction, so that $\boldsymbol{\mu} \cdot \mathbf{B} = 0$, $\mathbf{d} \cdot \mathbf{E} = 0$. Moreover, let there be no change in the dipoles along the external fields: $\mathbf{E} \cdot \nabla \mu = 0$, $\mathbf{B} \cdot \nabla d = 0$. After implying these conditions, insert (47) into (48) to obtain

$$\Phi^A(\theta) = \frac{i}{\hbar c} \oint (\boldsymbol{\mu} \times \mathbf{E} - \mathbf{d} \times \mathbf{B}) \cdot d\mathbf{r} + \frac{i}{2\hbar^2 c^2} \theta_{ab} \oint (\mathbf{k} + \boldsymbol{\mu} \times \mathbf{E} - \mathbf{d} \times \mathbf{B})_a \partial_b (\boldsymbol{\mu} \times \mathbf{E} - \mathbf{d} \times \mathbf{B}) \cdot d\mathbf{r}, \quad (50)$$

where $a, b = 1, 2, 3$. For $\mathbf{d} = 0$ the θ -deformation of the AC phase obtained in [6] follows

$$\Phi^{AC}(\theta) = \frac{i}{\hbar c} \oint (\boldsymbol{\mu} \times \mathbf{E}) \cdot d\mathbf{r} + \frac{i}{2\hbar^2 c^2} \theta_{ab} \oint (\mathbf{k} + \boldsymbol{\mu} \times \mathbf{E})_a \partial_b (\boldsymbol{\mu} \times \mathbf{E}) \cdot d\mathbf{r}. \quad (51)$$

For $\boldsymbol{\mu} = 0$, the HMW phase in noncommuting coordinates is obtained in accord with [7] as

$$\Phi^{HMW}(\theta) = -\frac{i}{\hbar c} \oint (\mathbf{d} \times \mathbf{B}) \cdot d\mathbf{r} - \frac{i}{2\hbar^2 c^2} \theta_{ab} \oint (\mathbf{k} - \mathbf{d} \times \mathbf{B})_a \partial_b (\mathbf{d} \times \mathbf{B}) \cdot d\mathbf{r}. \quad (52)$$

By putting (51) and (52) together

$$\Phi^{A1}(\theta) = \Phi^{AC}(\theta) + \Phi^{HMW}(\theta),$$

which means ignoring the terms behaving as μd in (50), the deformation of [8] follows¹. Although we used 3-dimensional vectors the formalism is effectively 2-dimensional because of the selected configurations leading to the AC and HMW phases.

The approach of [3] differs from the above formulation. In [3] one considers the θ -deformed Hamiltonian defined as the generalization of the one obtained in the uniform transverse magnetic field B . In terms of the related path integral one identifies

$$\tilde{A}_i(\theta, r) = \left(1 - \frac{e\theta B}{4\hbar c}\right)^{-1} A_i(r).$$

Then, one employs it in (48) with $\rho = -e/c$ to get the AB phase in noncommutative coordinates as

$$\Phi_{AB}^{II}(\theta) = -\frac{ie}{\hbar c} \left(1 + \frac{e\theta B}{4\hbar c}\right) \oint A_i(r) dr_i.$$

Now, let us present our approach following in part the receipt given in [3]. We deal with the configurations leading to vanishing scalar potentials, so that in general the Hamiltonian in noncommutative coordinates is written in terms of \hat{p} which is a realization of the algebra (13)–(16) as

$$H^{nc} = \frac{\hat{p}^2}{2m}. \quad (53)$$

Obviously, different realizations will lead to different Hamiltonians. Let (r_α, p_α) define the classical phase space variables corresponding to the operators $(r_\alpha^{op}, p_\alpha^{op} = -i\hbar\partial_\alpha)$. The classical Hamiltonian $H_{\text{eff}}(r, p)$ will be obtained from the related Hamiltonian operator in noncommutative space by substituting $p_\alpha^{op}, r_\alpha^{op}$ with the c-number variables p, r . To keep the discussion general let us define the classical θ -deformed Hamiltonian corresponding to (53) as

$$H_{\text{eff}}^{nc} = a_{\alpha\beta}(r, \theta) p_\alpha p_\beta + b_\alpha(r, \theta) p_\alpha + c(r, \theta), \quad (54)$$

¹There is a discrepancy of sign which seems due to a misprint in Eq. (33) of [8].

without specifying the coefficients $a_{\alpha\beta}(r, \theta)$, $b_\alpha(r, \theta)$ and $c(r, \theta)$. Plugging (54) into the path integral

$$Z = N \int d^d p d^d r \exp \left\{ \frac{i}{\hbar} \int dt [p^\alpha \dot{r}_\alpha - H_{\text{eff}}(p, r)] \right\}, \quad (55)$$

where N is the normalization factor, yields the partition function in the d -dimensional phase space:

$$Z = N \int d^d p d^d r \exp \left\{ \frac{i}{\hbar} \int dt [p^\alpha (\dot{r}_\alpha - b_\alpha(r, \theta)) - a_{\alpha\beta}(r, \theta) p_\alpha p_\beta - c(r, \theta)] \right\}.$$

Integration over the momenta gives the partition function in configuration space with the normalization factor N' as

$$Z = N' \int d^d r \exp \left\{ \frac{i}{\hbar} \int dt \left[\frac{1}{4} a_{\alpha\beta}^{-1}(r, \theta) (\dot{r}_\alpha - b_\alpha(r, \theta)) (\dot{r}_\beta - b_\beta(r, \theta)) - c(r, \theta) \right] \right\}.$$

This can be written as

$$Z = N' \int d^d r \exp \left\{ \frac{i}{\hbar} S + \frac{i}{\hbar} \int dr_\alpha \mathcal{A}^\alpha(r, \theta) \right\},$$

in terms of

$$S = \int dt \left[\frac{1}{4} a_{\alpha\beta}^{-1}(r, \theta) (\dot{r}^\alpha \dot{r}^\beta + b^\alpha(r, \theta) b^\beta(r, \theta)) - c(r, \theta) \right]$$

and the θ -deformed gauge field defined as

$$\mathcal{A}_\alpha(r, \theta) \equiv -\frac{1}{2} a_{\alpha\beta}^{-1}(r, \theta) b^\beta(r, \theta). \quad (56)$$

Hence, in general we can introduce the quantum phase as follows

$$\Phi = \frac{i}{\hbar} \oint \mathcal{A}^\alpha(r, \theta) dr_\alpha = -\frac{i}{2\hbar} \oint a_{\alpha\beta}^{-1}(r, \theta) b^\beta(r, \theta) dr_\alpha. \quad (57)$$

As the first specific example we would like to discuss the AB phase in noncommutative space adopting some different realizations. Hence, let the particles be confined to move on the $r_i = (x, y)$ plane, in the presence of an infinitely long, tiny solenoid placed along the z -axis. Obviously we set $\rho = -e/c$, moreover the nonvanishing components of θ and F are

$$\theta_{ij} = \epsilon_{ij} \theta, \quad F_{ij} = \epsilon_{ij} F_{12} = \begin{cases} \epsilon_{ij} B & \text{in,} \\ 0 & \text{out.} \end{cases}$$

Except on the solenoid, the conditions (19) are fulfilled, due to the fact that F_{12} is constant inside the solenoid and vanishes outside the solenoid. Thus, we are equipped with the realizations (17)–(18) and (22)–(23) in a consistent manner. We first deal with the realization given in (17) but ignore the e^2/c^2 terms, so that the related coefficients are

$$a_{ij}^{(1)}(r, \theta) = \frac{1}{2m} \left(1 - \frac{eF_{12}\theta}{2\hbar c} \right)^2 \delta_{ij}, \quad b_i^{(1)}(r, \theta) = \frac{e}{mc} \left(1 - \frac{eF_{12}\theta}{2\hbar c} \right) A_i. \quad (58)$$

The trajectory in (57) is chosen to enclose the origin, thus it yields

$$\Phi_{AB}^{nc(1)} = -\frac{ie}{\hbar c} \oint \left(1 + \frac{eF_{12}\theta}{2\hbar c} \right) A_i dr_i = -\frac{ie}{\hbar c} \left(1 + \frac{e\theta B}{2\hbar c} \right) \int \epsilon_{ij} \nabla_i A_j ds = \left(1 + \frac{e\theta B}{2\hbar c} \right) \Phi_{AB}, \quad (59)$$

where the AB phase is given in terms of $\Phi_0 = hc/e$ and the cross-sectional area of the solenoid S as

$$\Phi_{AB} = -2\pi i \frac{BS}{\Phi_0}.$$

When we consider (17) keeping the e^2/c^2 terms the coefficients become

$$a_{ij}^{(2)}(r, \theta) = \frac{1}{2m} \left(1 - \frac{eF_{12}\theta}{2\hbar c}\right)^2 \delta_{ij}, \quad b_i^{(2)}(r, \theta) = \frac{e}{mc} \left(1 - \frac{eF_{12}\theta}{2\hbar c}\right)^2 A_i. \quad (60)$$

Observe that (56) does not acquire any θ -deformation. As a result of this the phase is not deformed:

$$\Phi_{AB}^{nc(2)} = \Phi_{AB}. \quad (61)$$

For the realization (22) one can read the coefficients as follows

$$a_{ij}^{(3)}(r, \theta) = \frac{1}{2m} \left(1 - \frac{eF_{12}\theta}{\hbar c}\right)^2 \delta_{ij}, \quad b_i^{(3)}(r, \theta) = -\frac{eB}{2mc} \left(1 - \frac{eF_{12}\theta}{\hbar c}\right) \epsilon_{ij} r_j. \quad (62)$$

Hence, the θ -deformed AB phase is deduced as

$$\Phi_{AB}^{nc(3)} = \frac{ie}{2\hbar c} \oint \left(1 + \frac{eF_{12}\theta}{\hbar c}\right) F_{12} \epsilon_{ij} r_j dr_i = \left(1 + \frac{e\theta B}{\hbar c}\right) \Phi_{AB}, \quad (63)$$

where we used $S = \oint \epsilon_{ij} r_i dr_j / 2$.

Similar to the Hall conductivity (41), the realization (17) when the terms at the order of e^2/c^2 are kept, i.e. (60), does not procure any θ -deformation of the AB phase (61). However, the other realizations (58) and (62) led to (59) and (63) with different θ -dependent factors, in contrary to the Hall effect where they yielded the same factor as is given in (42). An approach to determine which formulation should be preferred is presented in the last section.

To discuss the AC, HMW and Anandan phases in noncommutative coordinates we will consider the realization (22) in 3 dimensions: $a, b = 1, 2, 3$. In general it leads to the θ -deformed gauge field (49), where

$$a_a^b = \frac{1}{2m} \left(\delta_a^b - \frac{2\rho}{\hbar} F_{ac} \theta^{cb} \right)$$

and

$$b_a = \frac{\rho}{2m} \left(F_{ab} - \frac{\rho}{\hbar} \theta_{ac} F^{cd} F_{db} \right) r^b.$$

As far as the conditions (19) are satisfied this construction is valid also for non-Abelian gauge fields. The θ -deformed phase factor is

$$\Phi^{nc} = -\frac{i\rho}{2\hbar} \oint \left(F^{ab} + \frac{\rho}{\hbar} F^{ac} F_{cd} \theta^{db} \right) r_a dr_b \quad (64)$$

Now we specify the gauge field as in (49) which is appropriate to discuss the AC, HMW and Anandan phases and consider the configuration: $\boldsymbol{\mu} = \mu \hat{z}$, $\mathbf{d} = d \hat{z}$; $\boldsymbol{\mu} \cdot \mathbf{B} = 0$, $\mathbf{d} \cdot \mathbf{E} = 0$ and $\mathbf{E} \cdot \nabla \mu = 0$, $\mathbf{B} \cdot \nabla d = 0$. Hence the problem is effectively 2-dimensional. The gauge field (49) is now Abelian and the nonvanishing components of the field strength are

$$F_{ij} = \epsilon_{ij} (-\mu \nabla \cdot \mathbf{E} + d \nabla \cdot \mathbf{B}). \quad (65)$$

Moreover, we consider the noncommutative plane by setting $\theta_{ij} = \epsilon_{ij}\theta$. As usual the electromagnetic fields are taken in the radial direction and their divergence vanish except in the infinitesimal regions around the origin where they satisfy[16]

$$\nabla \cdot \mathbf{E} = \frac{\lambda_e}{s'}, \quad \nabla \cdot \mathbf{B} = \frac{\lambda_m}{s''}.$$

We introduced s' and s'' which are, respectively, the areas of the infinitesimal regions where $\nabla \cdot \mathbf{E}$ and $\nabla \cdot \mathbf{B}$ are nonvanishing. Obviously, s' and s'' do not play any role in the original definition of the Anandan phase:

$$\Phi_A = \frac{\rho}{2} \oint F_{ij} r^i dr^j = -\frac{1}{\hbar c} (\mu \lambda_e - d\lambda_m). \quad (66)$$

The field strength(65) satisfies the conditions (19) so that we can use the realization leading to the phase (64). The Anandan phase in noncommutative space can be calculated as

$$\Phi_A^{nc} = \Phi_A \left[1 + \theta \left(\frac{\mu \lambda_e}{\hbar c s'} - \frac{d\lambda_m}{\hbar c s''} \right) \right], \quad (67)$$

where (66) is employed.

Imposing, respectively, $\lambda_e = 0$ and $\lambda_m = 0$ in (67), the noncommutative AC and HMW phases can be deduced as

$$\begin{aligned} \Phi_{AC}^{nc} &= \frac{d\lambda_m}{\hbar c} \left(1 - \theta \frac{d\lambda_m}{\hbar c s''} \right), \\ \Phi_{HMW}^{nc} &= -\frac{\mu \lambda_e}{\hbar c} \left(1 + \theta \frac{\mu \lambda_e}{\hbar c s'} \right). \end{aligned}$$

Let us compare the constructions of quantum phases in noncommutative coordinates given in [4]–[8] and the ones obtained here. We showed that the former formulations result from the θ -deformed gauge field given in (47) which is obtained from the Hamiltonian (46) by employing the eigenvalues of the kinetic momentum (45), so that one gets rid of the momentum operators p_α . Instead of eliminating them, what happens if one keeps the momentum operators p_α in the definition of the θ -deformed Hamiltonian? In this case, the resulting Hamiltonian would be written as

$$\tilde{H} = \frac{1}{2m} \left[\left(\delta_\alpha^\sigma - \frac{\rho}{2\hbar} \theta^{\beta\sigma} \partial_\beta A_\alpha(r) \right) p_\sigma - \rho A_\alpha(r) \right]^2. \quad (68)$$

Now, employing the corresponding classical Hamiltonian in the path integral (55) and integrating over the momenta would have resulted in the θ -deformed gauge field as in (56) which is momentum independent. In fact, the latter procedure is the one which we adopted, though our deformation procedure is different than employing the coordinate shift (44). Hence, interpretation of the θ -deformed terms in Hamiltonians as a contribution to the gauge field or to the kinetic term is the main difference between these approaches. In our formulation deformed phases are independent of the velocity of the scattered particles which is one of the main features of the original quantum phases.

5 Discussions

We discussed in detail the alternative method of establishing quantum mechanics in noncommutative coordinates which is applicable to dynamical systems whose observables are either functions of phase

space variables or matrices which may be independent of phase space coordinates. The alternative procedure itself leads to different deformed dynamical systems depending on the adopted representation of the deformed algebra (13)–(16) which is equivalent to identify the θ –deformed quantum phase space variables.

Within the alternative procedure we discussed the Hall effect in noncommutative coordinates in Section 3 and considered the quantum phases in noncommutative space in Section 4. Depending on to the realization adopted the resulting Hall conductivity as well as AB phase acquire diverse deformation factors in noncommutative coordinates. Although at first sight this may seem to be a pathological fact, as we will explain it is an embarrassment of riches permitting us to choose the realization adequate to the problem considered. One of the interpretations of the noncommutativity of coordinates is to consider it as an effective method of introducing interactions whose dynamical origins can be complicated[3, 17]. Once we determine which realization leads to the desired effective theory we can select to work within that representation. Let us illustrate this considering the integer quantum Hall effect. Demanding gauge invariance of the extended states yields quantization of the AB phase gained by the electrons in Hall effect which permits one to obtain the integer quantum Hall effect[18]. It is possible to extend this formulation to noncommutative space employing noncommutative versions of the Hall effect and the AB phase obtained in this work. Depending on the effective theory one desires to obtain by fixing the noncommutativity parameter θ in the deformed integer quantum Hall effect, one can select the appropriate deformations of the Hall effect and the AB phase. Then, the related Hamiltonian can be taken as the as the starting point of formulating a field theory which may be utilized to find out some other aspects. For example, we can employ the related noncommutative field theory to derive Green functions and obtain the quantized Hall effect in noncommutative coordinates similar to the ordinary case[19]. These are currently under inspection.

We clarified the relation and discrepancies between our approach and the existing works on defining the quantum phases in noncommutative spaces. We showed that in general quantum phases in noncommutative spaces either for Abelian or for non-Abelian gauge fields can be defined independent of the velocities of the particles, as the underlying commutative phases. We discussed the receipt on general grounds and in particular applied it to the noncommutative plane, due to the fact that configurations of the observed AB and AC quantum phases are effectively two-dimensional. Our results can also be applied to condensed matter systems like graphen which are effectively two-dimensional and where the quantum phases play an important role. This is an attractive problem because it may give some clues in testing the possible advantages of introducing noncommutative coordinates.

Obviously, we retained the first order contribution in θ and up to some few orders in coupling constants. However, our method provides a systematic receipt of deriving the higher contributions either in θ or in coupling constants. Moreover, introducing another deformation parameter similar to θ to render the momenta noncommutative even for vanishing $F_{\alpha\beta}$, is straightforward in our formulation.

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